THE FANO SURFACE OF THE FERMAT CUBIC THREEFOLD, THE DEL PEZZO SURFACE OF DEGREE 5 AND A BALL QUOTIENT.

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Abstract. We study in this paper a surface which has many intriguing and puzzling aspects: on one hand it is related to the Fano surface of lines of a cubic threefold, and the other hand it is related to a Ball quotient occurring in the realm of hypergeometric functions, as studied by Deligne and Mostow. It is moreover connected to a surface constructed by Hirzebruch, in its works for constructing surfaces with Chern Ratio equal to 3 by arrangements of lines on the plane. Furthermore, we obtain some results that are analogous to the results of Yamasaki-Yoshida when they computed the lattice of the Hirzebruch ball quotient surface.

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1. Introduction

Let us recall the following well known Theorem (see [8], Theorem 2) that relates an inequality between (log) Chern numbers with the theory of Ball quotients:

Theorem 1. *[Bogomolov, Hirzebruch, Miyaoka, Sakai, Yau]. Let S be a smooth projective surface with ample canonical bundle K and let D be a reduced simple normal crossing divisor on S (may be 0). Suppose that $K+D$ is nef and big. Then the following inequality:*

$$\bar{c}_1^2 \leq 3\bar{c}_2$$

holds, where $\bar{c}_1^2, \bar{c}_2$ are the logarithmic Chern numbers of $S' = S - D$ defined by: $\bar{c}_1^2 = (K+D)^2$ and $\bar{c}_2 = \chi_{top}(S')$. Furthermore, the equality occurs if and only if S' is a ball quotient i.e., if and only if we obtain S' by dividing the ball $\mathbb{B}_2$ with respect to a discrete group Γ of automorphisms acting on $\mathbb{B}_2$ properly discontinuously and with only isolated fixed points.
If a smooth projective surface X contains a rational curve and its Chern numbers satisfies $c_1^2 = 3c_2 > 0$, then X is the projective plane.

Few examples of surfaces with Chern ratio $\frac{c_1^2}{c_2}$ equals 3 have been constructed algebraically i.e. by ramified covers of known surfaces. The first examples are owed independently to Inoue and Livné (for a reference, see [1]). Among these examples, there is a surface $\mathbb{S}$, with Chern numbers:

$$c_1^2(\mathbb{S}) = 3c_2(\mathbb{S}) = 3^2 5^2,$$

that is the blow-down of (-1)-curves of a certain cyclic cover of the Shioda modular surface of level 5. Hirzebruch [5] then constructed other examples. Starting with the degree 5 del Pezzo surface $\mathcal{H}_1$ and for any $n > 1$, he constructed a cover $\eta_n : \mathcal{H}_n \to \mathcal{H}_1$ of degree n^5 branched exactly over the ten (-1)-curves of $\mathcal{H}_1$, with order n, such that for $n = 5$:

Theorem 2. *[Hirzebruch]. The Chern numbers of $\mathcal{H}_5$ satisfies:*

$$c_1^2(\mathcal{H}_5) = 3c_2(\mathcal{H}_5) = 3^2 5^4.$$

Then Ishida [7] established a link between the surface $\mathcal{H}_5$ and the Inoue-Livné surface $\mathbb{S}$:

Proposition 3. *[Ishida]. There is a étale map $\mathcal{H}_5 \to \mathbb{S}$ that is a quotient of $\mathcal{H}_5$ by an automorphism group of order 25.*

In the present paper, we give an example of a surface with log Chern numbers satisfying $\bar{c}_1^2 = 3\bar{c}_2$, and we obtain in part A) and B) of Theorem 4 below, results analogous to Theorem 2 and Proposition 3:
The variety that parametrizes the lines on a smooth complex cubic threefold $F \hookrightarrow \mathbb{P}^4$ is a smooth surface of general type called the Fano surface of F [3]. Let S be the Fano surface of the Fermat cubic threefold:

$$F = \{x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0\}.$$

It is the only Fano surface that contains 30 elliptic curves [11] ; the aim of this paper is to prove the following Theorem:

Theorem 4. *A) There is an open subvariety $S' \subset S$, complement of 12 disjoint elliptic curves on S, such that S' is a ball quotient with log Chern numbers $\bar{c}_1{}^2 = 3\bar{c}_2 = 3^4$.*
B) There is a étale map $\kappa : \mathcal{H}_3 \to S$ that is a quotient of $\mathcal{H}_3$ by an automorphism of order 3 and there is a degree 3^4 ramified cover $\eta : S \to \mathcal{H}_1$ branched with order 3 over the ten (-1)-curves of $\mathcal{H}_1$.
C) The surface $\mathcal{T} = \kappa^{-1}S' \subset \mathcal{H}_3$ is a ball quotient. Let Λ be the lattice of $\mathcal{T}$, i.e. the transformation group of the 2-dimensional unit ball $\mathbb{B}_2$ such that $\Lambda \setminus \mathbb{B}_2$ is isomorphic to $\mathcal{T}$. The lattice Λ is the commutator group of the congruence group:

$$\Gamma = \{T \in GL_3(\mathbb{Z}[\alpha]) / \, T \equiv I \; modulo \, (1-\alpha) \; and \; {}^t\bar{T}HT = H\}$$

where α is a primitive third root of unity, I is the identity matrix and H is Hermitian diagonal matrix with entries $(1, 1, -1)$ defining the 2-dimensional unit ball $\mathbb{B}_2$.
D) The lattice Γ is the Deligne-Mostow lattice associated to the 5-tuple $(1/3, 1/3, 1/3, 1/3, 2/3)$ (number 1 in [4] p. 86).

We wish to remark that in order to prove the part B) and C), we use a result of Namba on ramified Abelian covers of varieties that, to the best of our knowledge, has never been used before this paper.

We wish also to remark that parts C) and D) of Theorem 4 is the very analog of the following result of Yamazaki and Yoshida [14]:

Theorem 5. *[Yamazaki, Yoshida [14], Theorem 1]. The lattice Λ' of the ball quotient $\mathcal{H}_5$ is the commutator group of the congruence group:*

$$\Gamma' = \{T \in GL_3(\mathbb{Z}[\mu]) / \, T \equiv I \; modulo \, (1-\mu) \; and \; {}^t\bar{T}HT = H\}$$

where μ is a primitive fifth root of unity, I is the identity matrix and H is Hermitian diagonal matrix with entries $(1, 1, (1-\sqrt{5})/2)$, defining the 2-dimensional unit ball $\mathbb{B}_2$.
The lattice Γ' is the Deligne-Mostow lattice associated to the 5-tuple $(2/5, 2/5, 2/5, 2/5, 2/5)$ (number 4 in [4] p. 86).

Let us explain how the Deligne-Mostow lattices occur. Let $\mu = (\mu_1, \dots, \mu_5)$ be a 5-tuple of rational numbers with $0 < \mu_i < 1$ and $\sum \mu_i = 2$. Let d be the *l.c.m.* of the μ_i and let n_i be such that $\frac{n_i}{d} = \mu_i$. Let M be the moduli space of 5-tuples $x = (x_1, \dots, x_5)$ of distinct points on the projective line $\mathbb{P}^1$. For each point x of M, and $1 \leq i < j \leq 5$, we consider the periods:

$$\omega_{ij} = \int_{x_i}^{x_j} \frac{dz}{v}$$

on the curve $v^d = \Pi_{k=1}^{k=5}(z - x_i)^{n_k}$. These (multivalued) maps ω_{ij} clearly factor to $Q = M/Aut(\mathbb{P}^1)$ (that is isomorphic to the complement of the 10 (-1)-curves of the degree 5 del Pezzo surface). They are called hypergeometric functions and they satisfy what is called the Appell differential equations system. It turns out that the ω_{ij} span a 3 dimensional vector space W_μ, and these yield a multivalued holomorphic map $Q \to \mathbb{P}^2 = \mathbb{P}(W_\mu^*)$. In fact, the image of that map lies in the (copy of a) unit Ball $\mathbb{B}_2$ of $\mathbb{C}^2 \subset \mathbb{P}^2$. The multivaluedness is measured by the monodromy representation

$$\pi_1(Q) \to Aut(\mathbb{B}_2)$$

whose image is denoted by Γ_μ. The main results of the fundamental papers of Deligne and Mostow [4] and Mostow [9] are to prove that the group $\Gamma_\mu \subset PGL(W_\mu^*)$ is discontinuous and acts as a lattice on $\mathbb{B}_2$ for only a finite number of 5-tuple μ, to compute these μ and to provide examples of non-arithmetic lattices acting on $\mathbb{B}_2$.

2. The Fano surface of the Fermat cubic as a cover of $\mathcal{H}_1$.

Let S be the Fano surface of the Fermat cubic threefold:

$$F = \{x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0\} \hookrightarrow \mathbb{P}^4.$$

This surface is smooth, has Chern numbers $c_1^2 = 45$, $c_2 = 27$ and irregularity 5 (see [3], (0.7)). Let $A(3,3,5) \subset GL_5(\mathbb{C})$ be the group of diagonal matrices with determinant 1 whose diagonal elements are in $\mu_3 := \{x \in \mathbb{C}/x^3 = 1\}$. By Theorem 26 of [11], the automorphism group of F is the semi-direct product of the permutation group Σ_5 and $A(3,3,5) \simeq (\mathbb{Z}/3\mathbb{Z})^4$. An automorphism f of F preserves the lines and induces an automorphism on S denoted by $\rho(f)$. Let G be the group $\rho(A(3,3,5))$.

Let X be the quotient of S by the group G and let $\eta : S \to X$ be the quotient map.

Proposition 6. *The surface X is (isomorphic to) the del Pezzo surface $\mathcal{H}_1$, and the cover η is branched with index 3 over the ten (-1)-curves of X.*

Proof. Let us outline the proof of Proposition 6: using classical results on the quotient of a surface by a group action, we show that X is a smooth surface, then we compute its Chern numbers, and prove that the blowing down of four (-1)-curves on X is the plane, and that allows us to conclude that X is the degree 5 del Pezzo surface $\mathcal{H}_1$.

In order to prove that the surface X is smooth, we need to recall two lemmas: Let s be a point of S. Let us denote by $T_{S,s}$ the tangent space of S at s, by $L_s \hookrightarrow F$ the line on $F \hookrightarrow \mathbb{P}^4 = \mathbb{P}(\mathbb{C}^5)$ corresponding to s and by $P_s \subset \mathbb{C}^5$ the subjacent plane to the line L_s.

Lemma 7. *([11], Proposition 12). Let s be a fixed point of an automorphism $\rho(f)$ ($f \in A(3,3,5)$). The plane P_s is stable under the action of f and the eigenvalues of*

$$d\rho(f) : T_{S,s} \to T_{S,s}$$

are equal to the eigenvalues of the restriction of $f \in A(3,3,5)$ to the plane $P_s \subset \mathbb{C}^5$.

Hence this Lemma gives us the action of the differential $d\rho(f)$ on the fixed points of $\rho(f)$. Recall:

Lemma 8. *([11], Theorem 26). The Fano surface S of the Fermat cubic contains 30 elliptic curves, denoted by E_{ij}^{β} for indices $1 \leq i < j \leq 5$, $\beta \in \mu_3$. Each curve E_{ij}^{β} parametrizes the lines on a cone in the cubic F. Their configuration is as follows :*

$$E_{ij}^{\beta} E_{st}^{\gamma} = \begin{cases} 1 & \text{if } \{i,j\} \cap \{s,t\} = \emptyset \\ -3 & \text{if } E_{ij}^{\beta} = E_{st}^{\gamma} \\ 0 & \text{otherwise.} \end{cases}$$

The elements of the group G have order 3, for each of them it is easy to compute it closed set of fixed points. Let I be the set of points s in S such that s is an isolated fixed point of an element of G : I is the set of the 135 intersection points of the 30 elliptic curves. Let i,j,s,t be indices such that $\{i,j\} \cap \{s,t\} = \emptyset$ and let s be the intersection point of E_{ij}^{1} and E_{st}^{1}. The orbit of s by G is the set of the 9 intersection points of the curves E_{ij}^{β} and E_{st}^{γ}, $\beta, \gamma \in \mu_3$. Let be $s \in I$, the group:

$$G_s = \{g \in G / s \text{ is a isolated fixed point of } g\}$$

is isomorphic to μ_3^2 and, by Proposition 7, its representation on the space $T_{S,s}$ is isomorphic to the representation:

$$(\alpha_1, \alpha_2) \in \mu_3^2, \quad (\alpha_1, \alpha_2).(x,y) = (\alpha_1 x, \alpha_2 y) \in \mathbb{C}^2$$

on $\mathbb{C}^2$. That implies by [2], that the image of s is a smooth point of X, thus X is smooth.

The ramification index of $\eta : S \to X$ at the points of I is 9 and the ramification index of η on the curve E_{ij}^{β} is 3. Let us denote by K_V the canonical divisor of a surface V. Let be $\Sigma = \sum_{i,j,\beta} E_{ij}^{\beta}$; the ramification divisor of $\eta : S \to X$ is 2Σ and

$$K_S = \eta^* K_X + 2\Sigma.$$

By [3], Lemma 8.1 and Proposition 10.21, we know moreover that $\Sigma = 2K_S$, hence $3^4 (K_X)^2 = (\eta^* K_X)^2 = (-3K_S)^2 = 9.45$ and $(K_X)^2 = 5$.

The stabilizer in G of an elliptic curve $E_{ij}^{\beta} \hookrightarrow S$ contains 27 elements and the group that fixes each point of E_{ij}^{β} has 3 elements. Let $\eta_{ij}^{\beta} : E_{ij}^{\beta} \to X_{ij}$ be the restriction of η to E_{ij}^{β}. The curve X_{ij} is smooth because it is the quotient of a smooth curve by an automorphism group. The map η_{ij}^{β} is a degree 9 ramified cover over 3 points with ramification index 3, hence:

$$0 = \chi_{top}(E_{ij}^{\beta}) = 9(\chi_{top}(X_{ij}) - 3) + 3.3$$

and $\chi_{top}(X_{ij}) = 2$: X_{ij} is a smooth rational curve. As:

$$\eta^* X_{ij} = 3(E_{ij}^1 + E_{ij}^{\alpha} + E_{ij}^{\alpha^2}),$$

we deduce that the 10 curves X_{ij} have the following configuration:

$$X_{ij}X_{st} = \begin{cases} 1 & \text{if } \{i,j\} \cap \{s,t\} = \emptyset \\ -1 & \text{if } X_{ij} = X_{st} \\ 0 & \text{otherwise.} \end{cases}$$

Let $I' = \eta(I)$ and let $\Sigma' = \sum X_{ij}$. By additive property of the Euler characteristic, we have:

$$3^3 = \chi_{top}(S) = 3^4\chi_{top}(X - \Sigma') + 3^3\chi_{top}(\Sigma' - I') + 3^2\chi_{top}(I').$$

Moreover, $\chi_{top}(\Sigma') = 5$ (for an example of such computation see [12]) and we obtain $\chi_{top}(X) = 7$. We can blow-down four disjoint (-1)-curves among the ten curves X_{ij} and we obtain a surface with Chern numbers:

$$c_1^2 = 3c_2 = 9$$

but this surface contains 6 rational curves. Hence, by Theorem 1, it is the plane : X is the blow-up of the plane at four points. These points are in general position because of the intersection numbers of the X_{ij}, therefore X is the degree 5 del Pezzo surface $\mathcal{H}_1$ and the X_{ij} are its ten (-1)-curves.

We proved that the quotient map $\eta : S \to X = \mathcal{H}_1$ is an Abelian cover branched over the ten (-1)-curves of X with ramification index 3, and that completes the proof of Proposition 6. □

Let us now prove that:

Proposition 9. *There exists an étale map* $\kappa : \mathcal{H}_3 \to S$ *of degree* 3.

Proof. To prove Proposition 9, we begin to recall Namba's results on Abelian covers of algebraic varieties.

Let $D_1, \dots, D_s$ be irreducible hypersurfaces of a smooth projective variety M and let $e_1, \dots, e_s$ be positive integers. A covering $\pi : Y \to M$ is said to branch (resp. to branch at most) at $D = e_1D_1 + \cdots + e_sD_s$ if the branch locus is (resp. is contained in) $\cup D_i$ and the ramification index over D_i is e_i (resp. divides e_i). An Abelian covering $\pi : Y \to M$ which branches at D is said to be maximal if for every Abelian covering $\pi_1 : Y_1 \to M$ which branches at most at D, there is a map $\kappa : Y \to Y_1$ such that $\pi = \pi_1 \circ \kappa$. Let :

$$Div^0(M, D) = \{\hat{E} = \frac{a_1}{e_1}D_1 + \cdots + \frac{a_s}{e_s}D_s + E / a_i \in \mathbb{Z},\, E \,\text{integral},\, c_1(\hat{E}) = 0\}.$$

We say that $F_1, F_2 \in Div^0(M, D)$ are linearly equivalent $F_1 \sim F_2$ if $F_1 - F_2$ is integral and is a principal divisor. The following result is due to Namba:

Theorem 10. *(Namba [10], Thm. 2.3.18.). There is a bijective map of the set of (isomorphism classes of) Abelian coverings $\pi : Y \to M$ branched at most at D onto the set of finite subgroups $\mathcal{G}$ of $Div^0(M,D)/\sim$. Let G_π be the transformation group of the cover $\pi : Y \to M$. The bijective map satisfies:*
(1) $G_\pi \simeq \mathcal{G}(\pi)$
(2) let $\pi_1 : Y_1 \to M$ and $\pi_2 : Y_2 \to M$ be Abelian covers branched at most at D. There is a map $\kappa : Y_1 \to Y_2$ such that $\pi_1 = \pi_2 \circ \kappa$ if and only if $\mathcal{G}(\pi_1) \subset \mathcal{G}(\pi_2)$.

One applies this Theorem to the ten (-1)-curves X_{ij} of $X = \mathcal{H}_1$ with $e_i = 3$. The group $Div^0(\mathcal{H}_1, D)/\sim$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^5$ (for brevity, we skip the proof, but for an example of such a computation, see the proof of Lemma 13). As $\eta_3 : \mathcal{H}_3 \to \mathcal{H}_1$ is a degree 3^5 Abelian cover branched over the (-1)-curves with index 3, the group $\mathcal{G}(\eta_3)$ is equal to $Div^0(\mathcal{H}_1, D)/\sim$. Thus by Theorem 10, there exists $\kappa : \mathcal{H}_3 \to S$ such that $\eta_3 = \eta \circ \kappa$. As the maps η and η_3 are branched with order 3 over the ten (-1)-curves of $\mathcal{H}_1$, the map κ is étale. That completes the proof of Proposition 9. □

3. The Fano surface of the Fermat cubic as a ball quotient.

By [3], Theorem 7.8, (9.14) and (10.11), the Fano surface S of the Fermat cubic is smooth with invariants $c_1^2 = 45$ and $c_2 = 27$. Let $S' \subset S$ be the complement of the union D of 12 disjoints elliptic curves on S (there are 5 such sets of 12 elliptic curves, we can take by example the 12 curves E_{1i}^{β}, $2 \leq i \leq 5$, $\beta^3 = 1$). Let $\overline{c}_1^2, \overline{c}_2$ be the logarithmic Chern numbers of S'.

Proposition 11. *We have : $3\overline{c}_2 = \overline{c}_1^2 = 81$, therefore S' is a ball quotient.*

Proof. The canonical divisor K_S of S is ample, $K_S^2 = 45$ and $K_S E = -E^2 = 3$ for an elliptic curve $E \hookrightarrow S$ (see [3] (0.7) and [11] Proposition 10), therefore $K_S + D$ is nef. As $\bar{c}_1^2 = (K_S + D)^2 = 45 + 2.12.3 - 12.3 = 81$, the divisor $K_S + D$ is also big. As $\bar{c}_2(S') = e(S - D) = e(S) = 27$, we obtain $3\overline{c}_2 = \overline{c}_1^2$. Because D has no singularities, Theorem 1 implies that S' is a ball quotient. □

Let H be the Hermitian diagonal matrix with entries $(1, 1, -1)$ defining the 2-dimensional unit ball $\mathbb{B}_2$ into $\mathbb{P}^2$. Let α be a third primitive root of unity and let Γ be the congruence group:

$$\Gamma = \{T \in GL_3(\mathbb{Z}[\alpha])/\, T \equiv I \, \text{modulo} \, (1-\alpha) \, \text{and} \, {}^t\bar{T}HT = H\},$$

where I is the identity matrix. As κ is étale and S' is a ball quotient, the surface $\mathcal{T} = \kappa^{-1}S' \subset \mathcal{H}_3$ is a ball quotient. Let Λ be the transformation group of the 2-dimensional unit ball $\mathbb{B}_2$ such that $\Lambda \setminus \mathbb{B}_2 \simeq \mathcal{T}$. We have:

Theorem 12. *The group Λ is the commutator group of Γ.*

Proof. In order to compute Λ, we combine ideas in [14], where Yamazaki and Yoshida computed the lattice of the Ball quotient surface $\mathcal{H}_5$, and we use Namba's Theorem 10.

Let $\ell_1, \dots, \ell_6 \in H^0(\mathbb{P}^2, \mathcal{O}(1))$ be the linear forms defining the 6 lines on the plane going through 4 points in general position. Let $\mathcal{H}_3'$ be the normal algebraic surface determined by the field

$$\mathbb{C}(\mathbb{P}^2)((\frac{\ell_2}{\ell_1})^{1/3}, \dots, (\frac{\ell_2}{\ell_1})^{1/3}).$$

It is an Abelian cover $\pi : \mathcal{H}_3' \to \mathbb{P}^2$ of degree 3^5 of the plane branched with order 3 over the 6 lines $\{\ell_i = 0\}$ and the surface $\mathcal{H}_3$ is the fibered product of $\pi : \mathcal{H}_3' \to \mathbb{P}^2$ and the blow-up map $\tau : \mathcal{H}_1 \to \mathbb{P}^2$ (see [13] (1.3)). The situation is as follows:

$$\begin{array}{ccccc} & \overset{\kappa}{\swarrow} & \mathcal{H}_3 & \overset{\iota}{\to} & \mathcal{H}_3' \\ S & & \downarrow \eta_3 & & \downarrow \pi \\ & \underset{\eta}{\searrow} & \mathcal{H}_1 & \overset{\tau}{\to} & \mathbb{P}^2. \end{array}$$

We apply Namba's Theorem 10 to the 6 lines of the complete quadrilateral on the plane, with weights $e_i = 3$.

Lemma 13. *The group* $Div^0(\mathbb{P}^2, D)/\sim$ *is isomorphic to* $(\mathbb{Z}/3\mathbb{Z})^5$.

Proof. Let L_i be the line $\{\ell_i = 0\}$ and let L be a generic line. The group:

$$Div^0(\mathbb{P}^2, D)/\sim = \{aL + \sum \frac{a_i}{3} L_i / a, a_1, \dots, a_6 \in \mathbb{Z} \text{ and } 3a + \sum a_i = 0\}/\sim .$$

it is a sub-group of:

$$Div(\mathbb{P}^2, D)/\sim = \{aL + \sum \frac{a_i}{3} L_i / a, a_1, \dots, a_6 \in \mathbb{Z}\}/\sim,$$

where the rational divisor $E = aL + \sum \frac{a_i}{3} L_i$ in $Div(\mathbb{P}^2, D)$ is equivalent to 0 if and only if the a_i, $1 \leq i \leq 6$ are divisible by 3 and $c_1(E) = a + \frac{1}{3}\sum a_i = 0$ (here we use that linear and numerical equivalences are equal on the plane). The map :

$$\phi : \begin{cases} Div(\mathbb{P}^2, D)/\sim & \to & \mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^6 \\ aL + \sum \frac{a_i}{3} L_i & \to & (3a + \sum a_i, \bar{a}_1, \dots, \bar{a}_6) \end{cases}$$

is well defined and is an isomorphism. The group $Div^0(\mathbb{P}^2, D)/\sim$ is isomorphic to:

$$\{(a, \bar{a}_1, \dots, \bar{a}_6) \in \mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^6 / a = 0 \text{ and } \sum \bar{a}_i = 0\}$$

and is therefore isomorphic to $(\mathbb{Z}/3\mathbb{Z})^5$. □

By Lemma 13 and Theorem 10, the degree 3^5 Abelian cover $\pi : \mathcal{H}_3' \to \mathbb{P}^2$ is the maximal Abelian cover.

Let $b : \mathbb{P}^2 \to \mathbb{N}$ be the function such that $b(p) = 1$ outside the complete quadrilateral, $b(p) = 3$ on the complete quadrilateral minus the 4 triple points $p_1, \dots, p_4$, and $b(p) = \infty$ on these 4 points. The pair $(\mathbb{P}^2, b)$ is an orbifold (for the theory of orbifold we refer to [15], Chap. 5). By [6], Chap. 5, the universal cover of that orbifold is $\mathbb{B}_2$ with the transformation group Γ. Therefore, a cover $Z \to \mathbb{P}^2$ with branching index 3 over the complete quadrilateral corresponds to a normal sub-group K of Γ and Γ/K is isomorphic to the group of transformation of the covering $Z \to \mathbb{P}^2$.

If moreover, the cover $Z \to \mathbb{P}^2$ is Abelian, the group K contains the commutator group $[\Gamma, \Gamma]$, thus $\mathbb{B}_2/[\Gamma, \Gamma]$ is the maximal Abelian cover of $(\mathbb{P}^2, b)$. We have seen that the cover $\pi : \mathcal{H}'_3 \to \mathbb{P}^2$ of degree 3^5 is maximal among Abelian covers of $(\mathbb{P}^2, b)$, thus the lattice of the ball quotient $\mathcal{T}$ is the commutator group $[\Gamma, \Gamma]$. □

Moreover, we remark that :

Theorem 14. *The lattice* Γ *is the Deligne Mostow lattice number* 1 *in* [4] *p. 86.*

Proof. This is the fact that the universal cover of the orbifold $(\mathbb{P}^2, b)$ of the proof of Theorem 12 is $\mathbb{B}_2$ with the transformation group Γ.

□

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